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On the dynamics of a third order Newton's approximation method

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We provide an answer to a question raised by S. Amat, S. Busquier, S. Plaza on the qualitative analysis of the dynamics of a certain third order Newton type approximation function M_f, by proving that for functions f twice continuously differentiable and such that both f and its derivative do not have multiple roots, with at least four roots and infinite limits of opposite signs at $\pm\infty$, M_f has periodic points of any prime period and that the set of points a at which the approximation sequence $(M_f^n(a))_{n\in\mathbb{N}}$ does not converge is uncountable. In addition, we observe that in their Scaling Theorem analyticity can be replaced with differentiability.

1 Introduction

The classical Newton's Approximation Function $N_f(x) = x - f(x)/f'(x)$, for numerical approximation of roots of (nonlinear) functions f, under certain conditions of smoothness and distribution of roots and critical points, has a second order speed of convergence and, until now, it is considered as one of the most useful and reliable iterative method of this kind. S. Amat, S. Busquier, and S. Plaza, in [1], modified it to a third order approximation function $M_f(x) = N_f(x) - f(N_f(x))/f'(x)$ that is free of second derivatives and shows a remarkable robustness when compared to other methods. On the other hand, it is known that the classical Newton's Approximation Function N_f, when considered as a discrete dynamical system, shows chaotic behaviour, at least from two possible acceptions of the concept of chaos: first in the sense of T-Y. Li and J. A. Yorke [5], that is, existence of periodic points of any prime period, and second in the sense of R. Bowen [3], that is, a strictly positive topological entropy, as shown by M. Hurley and C. Martin [4], see also D. G. Saari and J. B. Urenko [7] for similar investigations.

In [1], the chaotic behaviour of M_f was numerically pointed out for polynomials of order less or equal than 3 by using a bifurcation diagram similar to that of the logistic map. They left open the question of performing a qualitative analysis on the discrete dynamical system associated to M_f in order to mathematically prove its chaotic behaviour. The main result of this article is Theorem 3.3 that shows that for functions f of Newton type, see Section 3 for definition, with at least four roots and infinite limits of opposite signs at $\pm\infty$, M_f has periodic points of any prime period and the set of points a at which the approximation sequence $\left(M_f^n(a)\right)_{n\in\mathbb{N}}$ does not converge is uncountable. For example, when considering polynomials, this result applies to all odd degree polynomials with a certain distribution of real roots. In view of Lemma 2.3, Theorem 3.3 might be extended to other classes of Newton's functions f having at least four roots and for which M_f has at least two bands that cover the whole real line $\mathbb{R}$, that is, there are two pairs of disjoint open intervals, formed by consecutive critical points of f, that are mapped by M_f onto the whole $\mathbb{R}$, see [4] for precise terminology.

In addition, in Theorem 4.1 we observe that in the Scaling Theorem from [1], which says that the dynamics of M_f is stable under affine conjugacy, as well as in its damped version as in [2], analyticity of f can be replaced by its differentiability. The Scaling Theorem is essential for the analysis performed in [1] because it

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reduces the study of the dynamics of M_f for a general class of functions f to the study of the dynamics of M_f for a considerably smaller class of simpler functions. For example, in order to understand the dynamics of M_f for quadratic polynomials f, it suffices to study only the dynamics of M_f for the quadratic polynomials $x^2,\ x^2+1,\ x^2-1$, while for cubic polynomials f, it suffices to study only the dynamics of M_f for the cubic polynomials $x^3,\ x^3+1,\ x^3-1,\ x^3+\gamma x+1$, with $\gamma \in \mathbb{R}$.

We might also add results on the lower estimation of the Bowen's topological entropy for M_f but, once Theorem 3.3 is obtained, these follow in an almost identical fashion as in [4], when considered the two kinds of bands induced by the critical points of f for M_f. Also, we observe that, for the class of functions that make the assumptions of Theorem 3.3, the damping with a parameter λ, cf. [2], does not change either the chaotic behaviour expressed as the existence of periodic points of any prime period or the uncountability of the set of all real numbers a for which the iterative sequences $\left(M^n_{\lambda,f}(a)\right)_{n=1}^{\infty}$ diverges.

2 Preliminaries

In this section we collect results related to periodic points of continuous real functions. We start by proving a sequence of lemmas that are essentially contained in [4] and more or less implicit/explicit in the works of N. A. Sharkovsky [8] and T.-Y. Li and J. A. Yorke [5].

The first lemma is a well known fixed point result and a direct consequence of the Intermediate Value Property for continuous functions.

Lemma 2.1 *If I and J are compact intervals and $f : I \to J$ is a continuous function with $f(I) \supseteq I$, then f has a fixed point.*

The next lemma is important for understanding the dynamics of continuous real functions, e.g. see Lemma 0 in [5]. We provide a proof for consistency.

Lemma 2.2 *Let J and K be nonempty compact intervals and let $f : J \to \mathbb{R}$ be a continuous function such that $f(J) \supseteq K$. Then, there exists a nonempty compact interval $L \subseteq J$ such that $f(L) = K$.*

Proof. Let $K = [a, b]$. If $a = b$ then we apply the Intermediate Value Theorem and get $c \in J$ such that $f(c) = a$ and let $L = [c, c]$, so let us assume that $a < b$. Since the set $\{x \in J \mid f(x) = a\}$ is compact and nonempty, there is a greatest element c in this set. If $f(x) = b$ for some $x \geq c$ with $x \in J$, then $x > c$ and letting d be the least of them, by the Intermediate Value Theorem $f([c, d]) = K$ and we let $L = [c, d]$. Otherwise, $f(x) = b$ for some $x < c$ and let c' be the largest of them. Then let d' be the smallest of the set of all $x > c'$ with $f(x) = a$ so that $[c', d']$ is an interval in J. Then $f([c', d']) = K$ and we let $L = [c', d']$. □

The main technical fact we use is a refinement of Lemma 2.2 in [4], implicit in the proof of Sharkovsky's Theorem [8], see also [5]. Recall that, a point $a \in M$ is called a *periodic point* for a function $g : M \to M$ if there exists $n \in \mathbb{N}$ such that $g^n(a) = a$, and the least $n \in \mathbb{N}$ with this property is called its *prime period*.

Lemma 2.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a function and let $I_1, I_2, \ldots, I_k$ be compact, disjoint and nondegenerate intervals, with $k \geq 2$, such that, for all $m \in \{1, 2, \ldots, k\}$, g is continuous on I_m and*

$$g(I_m) \supseteq \bigcup_{j=1}^{k} I_j.$$

Then:

(a) *For each $n \in \mathbb{N}$, g has at least $k(k-1)^{n-1}$ periodic points of prime period n, in particular, g has periodic points of any prime period.*

(b) *The set of all real numbers a for which the orbit $(g^n(a))_{n\in\mathbb{N}}$ makes a sequence that does not converge is uncountable.*

Proof. (a) For any $n \in \mathbb{N}$, take any sequence $(j_i)_{i=1}^n$ of length n with $j_i \in \{1, 2, \ldots, k\}$ and let $j_{n+1} = j_1$. Considering the sequence of intervals $(I_{j_i})_{i=1}^{n+1}$, by assumption,

$$g(I_{j_1}) \supseteq \bigcup_{j=1}^{k} J_j \supseteq \bigcup_{i=1}^{n} I_{j_i} \supseteq I_{j_2}.$$

By Lemma 2.2 there exists a compact interval $A_{j_1} \subseteq I_{j_1}$ such that $g(A_{j_1}) = I_{j_2}$. If $n = 1$, we observe that, since $g(A_{j_1}) = I_{j_2} = I_{j_1}$, by Lemma 2.1 it follows that g has a fixed point in the compact interval A_{j_1}.

If $n \geq 2$, then

$$g^2(A_{j_1}) = g(I_{j_2}) \supseteq \bigcup_{j=1}^{k} J_j \supseteq \bigcup_{i=1}^{n} I_{j_i} \supseteq I_{j_3},$$

and by Lemma 2.2 applied to g^2 we obtain a compact interval $A_{j_2} \subseteq A_{j_1} \subseteq I_{j_1}$ such that $g(A_{j_2}) = I_{j_3}$. Proceeding in a similar fashion, after n consecutive applications of Lemma 2.2, we obtain compact intervals

$$A_{j_n} \subseteq A_{j_{n-1}} \subseteq \cdots \subseteq A_{j_1} \subseteq I_{j_1},$$

such that

$$g^i(A_{j_i}) = I_{j_{i+1}}, \quad i = 1, \ldots, n, \tag{2.1}$$

in particular, $g^n(A_{j_n}) = I_{j_{n+1}} = I_{j_1}$. Then, by Lemma 2.1 there is a fixed point $a \in A_{j_n}$ for g^n, hence a is a periodic point for g of period n.

We observe now that, for $n = 1$ there are exactly k different choices for $j_1 = 1, \ldots, k$ and, since the intervals $J_1, \ldots, J_k$ are mutually disjoint, the k fixed points obtained, as explained before, are all different, hence g has at least k fixed points.

For $n \geq 2$, if $j_1 \neq j_i$ for all $i = 2, \ldots, n$, then the periodic point a for g obtained before from the sequence $(j_i)_{i=1}^n$ has prime period n. Indeed, for all $2 \leq i \leq n$, $a \in A_{j_i} \subseteq A_{j_{i-1}}$ and, by (2.1) it follows that $g^{i-1}(a) \in I_{j_i}$ hence, since $I_{j_1} \cap I_{j_i} = \emptyset$ it follows that a cannot have any period less than n.

We consider now two different sequences $(j_i)_{i=1}^n$, $(l_i)_{i=1}^n$ as before, hence there exists $r \in \{1, 2, \ldots, n\}$ such that $j_r \neq l_r$. If the fixed point a for the two sequences of intervals $(I_{j_i})_{i=1}^n$ and $(I_{l_i})_{i=1}^n$, obtained as before, is the same, then $(g^i(a))_{i=0}^n$, the orbit of a under g, is the same for both sequences. But then, $g^{r-1}(a) \in I_{j_r} \cap I_{l_r} = \emptyset$ hence we have a contradiction. Therefore, for the $k(k-1)^{n-1}$ different sequences $(j_i)_{i=1}^n$ of length n, formed with elements from the set $\{1, \ldots, k\}$ and subject to the condition $j_1 \neq j_i$ for all $i = 2, \ldots, n$, we have $k(k-1)^n$ different fixed points of prime period n.

(b) For any infinite sequence $(j_i)_{i=1}^\infty$ with elements from $\{1, 2, \ldots, k\}$, which is not eventually constant, by the same construction as above we get a sequence of nonempty compact intervals $(A_{j_i})_{i=1}^\infty$ subject to the properties

$$\cdots \subseteq A_{j_{i+1}} \subseteq A_{j_i} \subseteq \cdots \subseteq A_{j_2} \subseteq A_{j_1} \subseteq I_{j_1}, \tag{2.2}$$

and

$$g^i(A_{j_i}) = I_{j_{i+1}}, \quad i \in \mathbb{N}. \tag{2.3}$$

By the Finite Intersection Property we have

$$A = \bigcap_{i=1}^{\infty} A_{j_i} \neq \emptyset,$$

hence, any point $a \in A$ has the property that its orbit $(g^i(a))_{i=1}^\infty$ makes a sequence that does not converge. The sequence $(g^i(a))_{i=1}^\infty$ does not converge since, for all $i \in \mathbb{N}$ we have $g^i(a) \in I_{j_{i+1}}$, the sequence $(j_i)_{i=1}^\infty$ with elements from $\{1, 2, \ldots, k\}$ is not eventually constant, and the compact intervals $I_1, \ldots, I_k$ are mutually disjoint.

Let us consider two different sequences $(j_i)_{i=1}^\infty$ and $(l_i)_{i=1}^\infty$, formed with elements from the set $\{1, \ldots, k\}$ and not eventually constant. As before, to the sequence $(j_i)_{i=1}^\infty$ we associate the sequence of nonempty compact intervals $(A_{j_i})_{i=1}^\infty$ subject to the properties (2.2) and (2.3), and let $a \in \bigcap_{i=1}^\infty A_{j_i}$. Similarly, there is a sequence $(B_{l_i})_{i=1}^\infty$ of nonempty compact intervals subject to the properties

$$\cdots \subseteq B_{l_{i+1}} \subseteq B_{l_i} \subseteq \cdots \subseteq B_{l_2} \subseteq B_{l_1} \subseteq I_{j_1},$$

and

$$g^i(B_{l_i}) = I_{l_{i+1}}, \quad i \in \mathbb{N},$$

and let $b \in \bigcap_{i=1}^{\infty} B_{l_i} \neq \emptyset$. We claim that $a \neq b$. Indeed, there exists $r \in \mathbb{N}$ such that $j_r \neq l_r$, hence $g^r(a) \in I_{j_r} \cap I_{l_r} = \emptyset$, a contradiction.

In conclusion, there are as many real numbers a for which the sequence $(g^i(a))_{i=1}^{\infty}$ does not converge at least as many as sequences $(j_i)_{i=1}^{\infty}$ formed with elements from the set $\{1, \ldots, k\}$ and that are not eventually constant, and the latter set is uncountable. □

3 The dynamics of M_f

Following [4], the *Newton Class* is defined as the collection of all real functions f subject to the following conditions:

(nf1) f is of class $\mathcal{C}^2(\mathbb{R})$.
(nf2) If $f(x) = 0$ then $f'(x) \neq 0$.
(nf3) If $f'(x) = 0$ then $f''(x) \neq 0$.

Remarks 3.1 Let f be a Newton map.
(a) Clearly, both f and its derivative f' do not have multiple roots.
(b) The roots of f, and similarly the roots of f', do not have finite accumulation points. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct roots of f that accumulates to some $x_0 \in \mathbb{R}$ then, by the Interlacing Property of the roots of f and its derivative f', it follows that there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of distinct roots of f' converging to x_0. Since both f and f' are continuous, it follows that x_0 is a root for both f and f', contradiction with the statement at item (a). A similar argument shows that the roots of f' do not have finite accumulation points.

For a differentiable function f on an open set $D \subseteq \mathbb{R}$, the *Classical Newton's Approximation Function* N_f is the function defined for all $x \in D$ such that $f'(x) \neq 0$ by

$$N_f(x) = x - \frac{f(x)}{f'(x)}. \tag{3.1}$$

Following [1], the *Modified Newton's Approximation Function* M_f is the function, defined for all $x \in D$ such that $f'(x) \neq 0$ and $N_f(x) \in D$,

$$M_f(x) = x - \frac{f(x)}{f'(x)} - \frac{f(x - \frac{f(x)}{f'(x)})}{f'(x)} \tag{3.2}$$

or, in terms of N_f

$$M_f(x) = N_f(x) - \frac{f(N_f(x))}{f'(x)}. \tag{3.3}$$

The following lemma provides some information on the behavior of N_f in the neighbourhood of the critical points of a Newton map f, see Remark 1.3 in [4].

Lemma 3.2 *Let f be a Newton map and let c_1 and c_2 be two consecutive roots of f' such that in the interval (c_1, c_2) there is a unique root of f. Then*

$$\lim_{x\to c_1+} N_f(x) = -\lim_{x\to c_2-} N_f(x) = \pm\infty.$$

P r o o f . Indeed, near c_1+ and c_2-, f has opposite signs since it has a unique root inside the interval (c_1, c_2), while f' does not change its sign and goes to zero when approaching both c_1+ and c_2-. Since the term x is majorised by the term $\frac{f(x)}{f'(x)}$ near c_1+ and c_2-, we have the result. □

Here is the main result that shows that the Modified Newton's Approximation Function M_f provides chaotic behaviour for a large class of Newton's functions f.

Theorem 3.3 *Let the function $f : \mathbb{R} \to \mathbb{R}$ have the following properties:*

(i) *f is a Newton's function.*
(ii) $\lim_{x\to+\infty} f(x) = -\lim_{x\to-\infty} f(x) = \pm\infty$.
(iii) *f has at least four real roots.*

Then:
(a) *M_f has periodic points of any prime period.*
(b) *The set of all real numbers a for which the sequence $(M_f^n(a))_{n\in\mathbb{N}}$ does not converge is uncountable.*

Proof. By Rolle's Theorem, between any two consecutive roots of f there exists a root of its derivative f' hence, by Remarks 3.1 and property (iii), we can choose four consecutive roots $r_1 < r_2 < r_3 < r_4$ and three or four roots $c_1 < c_2 \leq c_2' < c_3$ of f' in the intervals (r_i, r_{i+1}) for $i = 1, 2, 3$, respectively, with a unique root of f inside each of the intervals (c_1, c_2), (c_2', c_3) and with no other root of f' there. With this choice, f' does not change its sign on the intervals (c_1, c_2) and (c_2', c_3). By using Lemma 3.2 and property (ii), we have

$$\lim_{x\to c_1+} f(N_f(x)) = -\lim_{x\to c_2-} f(N_f(x)) = \pm\infty. \tag{3.4}$$

In the following we show that

$$\lim_{x\to c_1+} M_f(x) = -\lim_{x\to c_2-} M_f(x) = \pm\infty. \tag{3.5}$$

Indeed, when $x \to c_1+$, the first equality will follow if we show that

$$\lim_{x\to c_1+} = \frac{-f(x) - f(N_f(x))}{f'(x)} = \pm\infty,$$

and, taking into account that $\lim_{x\to c_1+} f(x) = f(c_1) \in \mathbb{R}$, we observe that the latter will follow if we prove that

$$\lim_{x\to c_1+} \frac{-f(N_f(x))}{f'(x)} = \pm\infty,$$

which actually follows from (3.4) and the fact that f' does not change its sign in the interval (c_1, c_2). Hence, the first equality in (3.5) is proven. The fact that $\lim_{x\to c_2-} M_f(x) = \pm\infty$ is proven similarly, the only thing that remains to be shown is that the two limits in (3.5) have different signs, which actually is a consequence of (3.4) and the fact that f' has constant sign on (c_1, c_2). Therefore, (3.5) is proven.

Similarly, we have that

$$\lim_{x\to c_2'+} M_f(x) = -\lim_{x\to c_3-} M_f(x) = \pm\infty.$$

Therefore, there exists ε sufficiently small such that, letting $I_1 = [c_1 + \varepsilon, c_2 - \varepsilon]$ and $I_2 = [c_2' + \varepsilon, c_3 - \varepsilon]$ we have $M_f(I_j) \supseteq [c_1, c_3]$, for $j = 1, 2$, hence $M_f(I_1)$ and $M_f(I_2)$ contain $I_1 \cup I_2$. Finally, Lemma 2.3 is now applicable with $k = 2$, which finishes the proof. □

Remark 3.4 In order to reduce the chaotic behaviour and improve numerical parameters of approximation for lower order polynomials, in [2] and [6] damped Newton's methods have been considered. More precisely, letting λ be the damping parameter, one defines $N_{\lambda,f}$ and $M_{\lambda,f}$ as follows:

$$N_{\lambda,f}(x) = x - \lambda \frac{f(x)}{f'(x)}, \tag{3.7}$$

and

$$M_{\lambda,f}(x) = N_{\lambda,f}(x) - \lambda \frac{f(N_{\lambda,f}(x))}{f'(x)}. \tag{3.8}$$

It is easy to observe, by inspection, that Lemma 3.2 and Theorem 3.3 remain true if $N_{\lambda,f}$ and $M_{\lambda,f}$ replace N_f and, respectively, M_f, for arbitrary damping parameter $\lambda > 0$, hence the chaotic behaviour characterised by existence of periodic points of any prime period, as well as the uncountability of the set of points of divergence of iteration of M_f, remain unaltered by damping, for the class of functions considered in Theorem 3.3.

4 The Scaling Theorem

In this section we observe that the Scaling Theorem, cf. Theorem 1 in [1], which says that M_f is stable under affine conjugation, remains true if we replace the analyticity condition on the function f with its differentiability.

Theorem 4.1 (The Scaling Theorem) *Let f be a differentiable function on $\mathbb{R}$. Let $T(x) = ax + b$ be an affine map with $a \neq 0$. Then, for all $x \in \mathbb{R}$ with $f'(x) \neq 0$, we have*

$$\big(T \circ M_{f\circ T} \circ T^{-1}\big)(x) = M_f(x).$$

We first prove a lemma saying that the transformation N_f satisfies a similar property of stability under linear conjugation.

Lemma 4.2 *Under the assumptions as in Theorem 4.1, for all $x \in \mathbb{R}$ with $f'(x) \neq 0$, we have*

$$\big(T \circ N_{f\circ T} \circ T^{-1}\big)(x) = N_f(x).$$

Proof. Clearly, for all $y \in \mathbb{R}$ we have $(f \circ T)'(y) = af'(ay + b)$ hence, if $f'(ay + b) \neq 0$ we have

$$N_{f\circ T}(y) = y - \frac{f(ay+b)}{af'(ay+b)}, \tag{4.1}$$

and then

$$\begin{aligned}\big(T \circ N_{f\circ T}\big)(y) &= ay - a\frac{f(ay+b)}{af'(ay+b)} + b\\ &= ay + b - \frac{f(ay+b)}{f'(ay+b)}\\ &= N_f(ay+b) = (N_f \circ T)(y).\end{aligned}$$

Since $T^{-1}(x) = \frac{x-b}{a}$, letting $x = ay + b$ we get

$$\big(T \circ N_{f\circ T} \circ T^{-1}\big)(x) = N_f(x).$$

□

Proof of Theorem 4.1 By (3.2), for arbitrary $y \in \mathbb{R}$ with $f'(ay + b) \neq 0$, from (4.1) we obtain

$$\big(T \circ M_{f\circ T}\big)(y) = aN_{f\circ T}(y) + b - \frac{f(aN_{f\circ T}(y) + b)}{f'(ay+b)},$$

and then, letting $x = ay + b$,

$$\big(T \circ M_{f\circ T} \circ T^{-1}\big)(x) = \big(T \circ N_{f\circ T} \circ T^{-1}\big)(x) - \frac{f(\big(T \circ N_{f\circ T} \circ T^{-1}\big)(x))}{f'(a(\frac{x-b}{a}) + b)}$$

whence, applying Lemma 4.2,

$$= N_f(x) - \frac{f(N_f(x))}{f'(x)} = M_f(x).$$

□

Remark 4.3 The Scaling Theorem remains true, with almost exactly the same proof, for the damped Newton's function $M_{\lambda,f}$, see (3.7), for arbitrary damping parameter $\lambda \neq 0$, more precisely, we have

$$\big(T \circ M_{\lambda,f\circ T} \circ T^{-1}\big)(x) = M_{\lambda,f}(x),$$

for all $x \in \mathbb{R}$ such that $f'(x) \neq 0$. For the case of the damped Newton's function $N_{\lambda,f}$, see (3.6), the corresponding generalisation of Lemma 4.2,

$$\big(T \circ N_{\lambda,f\circ T} \circ T^{-1}\big)(x) = N_{\lambda,f}(x),$$

follows from the proof of Theorem 2.1 in [6]; although it was stated for analytic functions f, the assumption was not used in that proof.

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